CCRT: Categorical and Combinatorial Representation Theory.

From combinatorics of universal problems
to usual applications.

G.H.E. Duchamp
Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
Karol A. Penson, Darij Grinberg, Hoang Ngoc Minh, C. Lavault,
C. Tollu, N. Behr, V. Dinh, C. Bui,
Q.H. Ngô, N. Gargava, S. Goodenough.

CIP seminar,

Friday conversations:

For this seminar, please have a look at Slide CCRT[n] & ff.

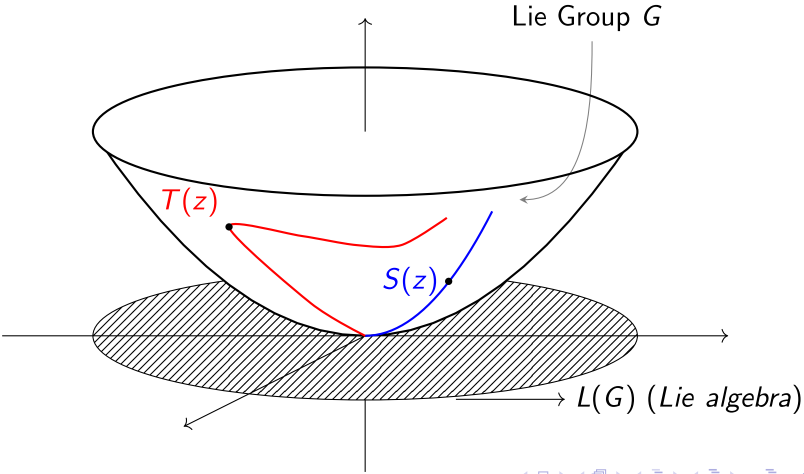
Goal of this series of talks

The goal of these talks is threefold

1. Category theory aimed at "free formulas" and their combinatorics
2. How to construct free objects
    1. w.r.t. a functor with - at least - two combinatorial applications:
        1. the two routes to reach the free algebra
        2. alphabets interpolating between commutative and non commutative worlds
    2. without functor: sums, tensor and free products
    3. w.r.t. a diagram: limits
3. Representation theory: Categories of modules, semi-simplicity, isomorphism classes i.e. the framework of Kronecker coefficients.
4. MRS factorisation: A local system of coordinates for Hausdorff groups.

CCRT[10]: Lie-theoretic aspects of Noncommutative Differential Equations.

1. We start from the picture of last friday (with two paths drawn)



2. When one sees the following

Proposition

i) Series $S_{Pic}^{z_0}$ is the unique solution of

$$\left\{\begin{array}{rcl} \mathbf{d}(S) & = & M.S \text{ with } M \in \mathcal{H}(\Omega)_+\langle\langle X\rangle\rangle \qquad (HNCDE) \\ S(z_0) & = & 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} \end{array}\right. \tag{1}$$

ii) The set of solutions of $\mathbf{d}(S) = M.S, (HNCDE)$ is $S_{Pic}^{z_0}.\mathbb{C}\langle\langle X\rangle\rangle$.

ii) The complete set of solutions of $(HNCDE + \langle S|1\rangle = 1)$ is then $S_{Pic}^{z_0}.(1 + \mathbb{C}_+\langle\langle X\rangle\rangle)$ (the NC Galois group is then in red).

3. Here Picard's process is defined by

The series $S_{Pic}^{z_0}$ $(z_0 \in \Omega)$ can be computed by Picard's process

$$S_0 = 1_{X^*} \ ; \ S_{n+1} = 1_{X^*} + \int_{z_0}^{z} M.S_n \tag{2}$$

and its limit is $S_{Pic}^{z_0} := \lim_{n\to\infty} S_n \ (= \sum_{w\in X^*} \alpha_{z_0}^{z}(w)\, w)$.

4. and the following

Theorem (Analyse et Géometrie, Cargèse, IESC, 21-24 Nov. 2017)

Let

$$(TSM) \qquad \mathbf{d}S = M_1 S + S M_2 \ . \qquad (3)$$

with $S \in \mathcal{H}(\Omega)\langle\langle X \rangle\rangle,\ M_i \in \mathcal{H}(\Omega)_+\langle\langle X \rangle\rangle$

(i) *Solutions of* (TSM) *form a* $\mathbb{C}$*-vector space.*

(ii) *Solutions of* (TSM) *have their constant term (as coefficient of* 1_{X^*}*) which are constant functions (on* Ω*); there exists solutions with constant coefficient* 1_Ω *(hence invertible).*

(iii) *If two solutions coincide at one point* $z_0 \in \Omega$ *(or asymptotically), they coincide everywhere.*

5. ... one cannot prevent thinking about Lie theory.
Let us take a look there.

G	$L(G)$	Cat	Eqns	Char=1
$U(n)$	$dX + dX^* = 0$	$\mathbb{R}$	$X(X^*) = I$	-
$SU(n)$	$dX + dX^* = 0$ $tr(X) = 0$	$\mathbb{R}$	$X(X^*) = I$	$det(X)$
$GL(n, \mathbf{k})$	$\mathbf{k}^{n \times n}$	$\mathbb{R}, \mathbb{C}$	$det(X) \neq 0$	-
$SL(n, \mathbf{k})$	$tr(X) = 0$	$\mathbb{R}, \mathbb{C}$	$det(X) = 1$	or $det(X)$
$Mag(\mathbf{k}, X)$	$\mathbf{k}_+\langle\langle X\rangle\rangle$	$\mathbb{Q} \subset \mathbf{k}$	$\epsilon(S) = 1$	-
$Haus(\mathbf{k}, X)$	$\Xi_{inf}(\mathbf{k}, X)$	$\mathbb{Q} \subset \mathbf{k}$	$\Delta_{\text{Ш}}(S) = S \otimes S$	$\langle S \vert 1_{X^*}\rangle = 1$

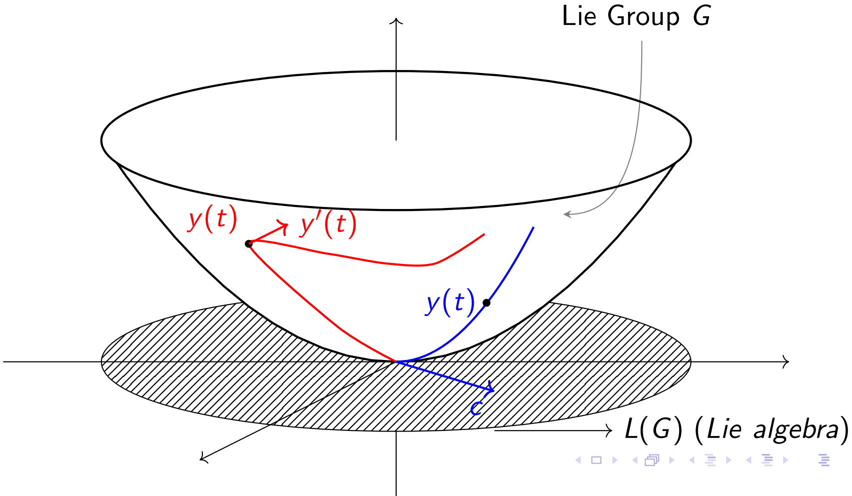
6. We now take advantage of some simple facts about Lie algebras
   1. There is a local Log-Exp mutually inverse correspondence (in the formal - unipotent - world it will global)
   2. If a C^1 path $t \mapsto \gamma(t)$ is drawn on G,
      i) for each t_0, $\gamma'(t_0) \in T_{\gamma(t_0)}$ and then
      ii) $m(t) = \gamma'(t)(\gamma(t)^{-1})$ is C^0 and drawn on $L(G)$.
   3. Then γ is a solution of the system

$$\begin{cases} y' & = & m(t).y \\ y(t_0) & = & y_0 \end{cases} \tag{4}$$

   4. Conversely "if a C^1 path γ is a solution of the system (4) (with $m(t)$ C^0 drawn on $L(G)$ and $y_0 \in G$), then γ is drawn on G" through Poincaré-Hausdorff formula.
   5. The proof of this converse holds for (closed) subgroups of invertible in Banach algebras and
   6. One-parameter groups (OPG) are obtained with $m(t) = c \in L(G),\ t_0 = 0,\ y_0 = 1$ (precisely this one is $\exp(c.t)$ i.e. the OPG with infinitesimal generator c).
   7. We will return to OPG later. For now, let us focus on general paths in the Noncommutative realm.

Every path drawn on the group is a solution of

$$y'(t) = m(t)y(t)$$



Examples (Lie-group side)

1. The Lie algebra of $SU(n)$ ($L(SU(n)) = \mathfrak{su}(n)$) consists of $n \times n$ skew hermitian traceless complex matrices (see table in slide 6). For example

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} i\,a & -\bar{z} \\ z & -i\,a \end{pmatrix} :\; a \in \mathbb{R}, z \in \mathbb{C} \right\} \tag{5}$$

2. Therefore a basis of its Lie algebra is

$$u_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{6}$$

and, for C^1 functions (i.e. in some $C^1(]a,b[,\mathbb{R})$) $f_i,\ i = 1..3$ the path $\gamma(t) = e^{f_1(t).u_1} e^{f_2(t).u_2} e^{f_3(t).u_3}$ is drawn on $SU(2)$.

3. Then, using conjugations, one can calculate explicitely the left multiplier $m(t)$ of $\gamma(t)$ i.e. $m(t)$ such that $\gamma'(t) = m(t)\gamma(t)$ (left to the reader so far).

Formal side

7. One can use Picard's process to construct solutions of NCDE (3) in slide 5 (and this will be generalized to other monoids). Doing this, one obtains C^ω-paths drawn on the Magnus group

$$Mag(\mathbb{C},X) = 1 + \mathbb{C}_+\langle\langle X\rangle\rangle$$

8. It suffices to modify this process by

$$S_0 = 1_{X^*}\ ;\ S_{n+1} = 1_{X^*} + \int_{z_0}^{z} M_1.S_n + S_n.M_2 \tag{7}$$

9. and the limit S (which is easily proved to exist as $\langle M_i|1_{X^*}\rangle = 0$) is the unique solution of

$$\left\{\begin{array}{rcl} \mathbf{d}(S) & = & M_1.S + S.M_2 \\ S(z_0) & = & 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} \end{array}\right. \tag{8}$$

10. Remark that S is solution of an equation $S' = MS$ as it is drawn on $Mag(\mathbb{C},X) = 1 + \mathbb{C}_+\langle\langle X\rangle\rangle$.

An application to renormalization

11. One can construct, using improper integrals the solution G_0 of the following system (with asymptotic initial condition)

$$\left\{ \begin{array}{rcl} \mathbf{d}(S) & = & (\frac{x_0}{z} + \frac{x_1}{1-z}).S \\ \lim_{z\to 0} S.e^{-x_0 log(z)} & = & 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} \end{array} \right. \qquad (9)$$

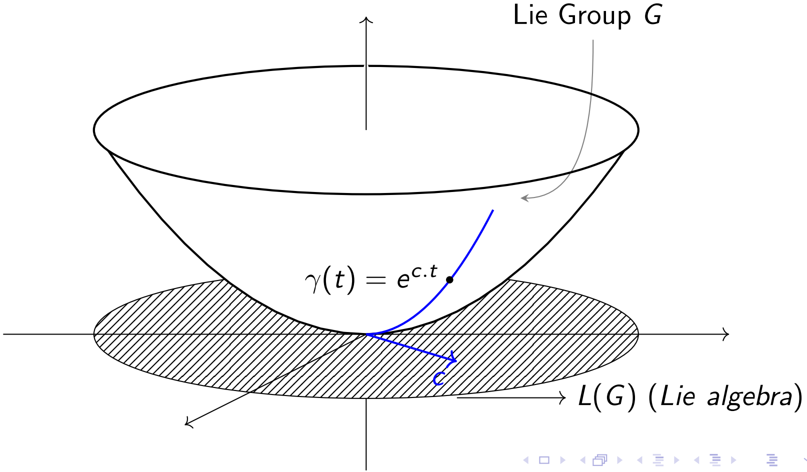
12. It then interesting to study $T = G_0.e^{-x_0 log(z)}$ which satisfies the two sided evolution equation (TSM)

$$\mathbf{d}(T) = (\frac{x_0}{z} + \frac{x_1}{1-z}).T + T.(-\frac{x_0}{z})$$

13. Next, one proves that T is group-like, factorizes through (MRS) and that $\lim_{z\to 0} T(z) = 1$.
14. We now embark to exponentials (OPG), MRS and Wei-Norman theorem.

Exponentials (OPG)

15. Here a one-parameter group (OPG) with infinitesimal generator (i.e. tangent vector at the origin). These OPG are also geodesics for every left-invariant Riemanian structure.



Product of exponentials: Wei-Norman theorem

16. We have the following theorem (true for all **k**-Lie group). See also [3] ch III §8 Ex.4 and Mathoverflow question "Local coordinates on infinite dimensional Lie groups and factorization of Riemann polylogarithms".
https://mathoverflow.net/questions/203771

Theorem (Wei-Norman theorem)

*Let G be a **k**-Lie group (of finite dimension) ($\mathbf{k} = \mathbb{R}$ or $k = \mathbb{C}$) and let $\mathfrak{g}$ be its k-Lie algebra. Let $B = \{b_i\}_{1\leq i\leq n}$ be a (linear) basis of it. Then, there is a neighbourhood W of 1_G (within G) and n analytic functions (local coordinates)*

$$W \to k,\ (t_i)_{1\leq i\leq n}$$

such that, for all $g \in W$

$$g = \prod_{1\leq i\leq n}^{\rightarrow} e^{t_i(g)b_i} = e^{t_1(g)b_1} e^{t_2(g)b_2} \dots e^{t_n(g)b_n}.$$

Towards the formal realm: classical construction/1

All definitions of algebra (resp. large algebra) of a monoid, Lie algebra, enveloping algebra, used here are standard and can be taken e.g. from [1, 3] (I can go into detail interactively on request by email).
Let X be a set (of variables, or indeterminates, or an alphabet), k a $\mathbb{Q}$-algebra and let

$$k\langle X\rangle, k\langle\langle X\rangle\rangle, \mathcal{L}_k\langle X\rangle, \mathcal{L}_k\langle\langle X\rangle\rangle$$

be respectively the free algebra (i.e. the algebra of noncommutative polynomials or the algebra of the free monoid X^*), the algebra of noncommutative formal power series (i.e. the large algebra of the free monoid X^*) see [1], the free Lie algebra and the Lie algebra of Lie series [3]. We will use the natural pairing between $k\langle\langle X\rangle\rangle = k^{X^*}$ and $k\langle X\rangle = k^{(X^*)}$ given by the following sum on the words

$$\langle S|P\rangle = \sum_w coeff(S,w)coeff(P,w)$$

Towards the formal realm: classical construction/2

It is well known that

$$k\langle X\rangle = \mathcal{U}(\mathcal{L}_k\langle X\rangle)\ .$$

As such, it admits a structure of Hopf algebra

$$(k\langle X\rangle, conc, 1_{X^*}, \Delta_{shuffle}, \epsilon, S)$$

conc being the concatenation, $\Delta_{shuffle}$ being the dual law of the shuffle product, $\epsilon(P) = \langle P|1_{X^*}\rangle$ (constant term) and $S(a) = -a$ for all $a \in X$; Every basis $(B = (b_i)_{i\in I};\ I$ totally ordered) of $\mathcal{L}_k\langle X\rangle$ (which is free, for all rings k) can be extended to a Poincaré-Birkhoff-Witt basis of $k\langle X\rangle$, parametrized by the multiindices of $\mathbb{N}^{(I)}$. The multi-index product is defined as follows. For every $\alpha \in \mathbb{N}^{(I)}$, we set

$$B^\alpha = b_{i_1}^{\alpha_1} b_{i_2}^{\alpha_2} \cdots b_{i_m}^{\alpha_m}$$

with $supp(\alpha) = \{i_1 < i_2 < \cdots i_m\}$.

Towards the formal realm: classical construction/3

Now, if B is multi-homogeneous (w.r.t. the $\mathbb{N}^{(X)}$-grading), so is $(B^\alpha)_{\alpha\in\mathbb{N}^{(I)}}$ and there is a unique family of polynomials B_α such that

$$\langle B_\alpha | B^\beta \rangle = \delta_{\alpha,\beta} \qquad \text{(Dual -- Basis)}$$

Now within the algebra of double series (whose support is $k^{X^*\otimes X^*}$ endowed with the law *shuffle*$\hat{\otimes}$*conc*, M.P. Schützenberger (see [3,4]) gave the beautiful formula

$$\sum_{w\in X^*} w\hat{\otimes}w = \prod_{i\in I}^{\rightarrow} e^{B_{e_i}\hat{\otimes}b_i} \qquad (10)$$

where e_i are the irreducibles of the monoid $\mathbb{N}^{(I)}$ defined by $e_i(j)=\delta_{i,j}$ (in particular $B^{e_i}=b_i$). This can be used to provide a system of local coordinates on the *Hausdorff group* (this is the closed subgroup of the Magnus group of primitive series).

Towards the formal realm: classical construction/4

$$Haus_k(X) = \{e^L\}_{L \in \mathcal{L}_k\langle\langle X \rangle\rangle} = \{S \in k\langle\langle X \rangle\rangle | \epsilon(S) = 1,\ \Delta_{shuffle}(S) = S \hat{\otimes} S\}$$

because, in this case, $S \otimes Id$ is compatible with the law of the double algebra and then, applying this operator to (10), we get

$$S = (S \hat{\otimes} Id)(\sum_{w \in X^*} w \hat{\otimes} w) = \prod_{i \in I}^{\rightarrow} e^{\langle S | B_{e_i} \rangle\, b_i}$$

which is a system of local coordinates for the group $Haus_k(X)$.

Towards the formal realm: classical construction/4

Application to Riemann zeta functions. –
When one multiplies several zeta values

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

multi-zeta values do appear, they are defined by

$$\zeta(s_1, s_2, \cdots s_k) = \sum_{n_1 > n_2 > \cdots n_k \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}} \ . \tag{11}$$

Towards the not-so-formal realm

When $s_1, s_2, \cdots s_k$ are integers, the link with the shuffle product is that the quantity (11) converges when $s_1 > 1$ and, coding $(s_1, s_2, \cdots s_k)$ by the word $w = (x_0^{s_1-1}x_1x_0^{s_1-1}x_1 \cdots x_0^{s_k-1}x_1)$ (here $X = \{x_0, x_1\}$)
and recoding (11) by $\tilde{\zeta}(w) = \zeta(s_1, s_2, \cdots s_k)$ one can prove that $\tilde{\zeta}$ can be extended uniquely as a shuffle character of $\mathbb{Q}\langle X\rangle$ satisfying
$\tilde{\zeta}(x_0) = \tilde{\zeta}(x_1) = 0$ so that, applying (11) we get

$$\tilde{\zeta} = (\tilde{\zeta}\hat{\otimes}Id)(\sum_{w\in X^*} w\hat{\otimes}w) = \prod_{i\in I}^{\rightarrow} e^{\tilde{\zeta}(B_{e_i})\, b_i} \tag{12}$$

for every multihomogeneous basis B of the free Lie algebra $\mathcal{L}_{\mathbb{Q}}\langle X\rangle$.

Towards the formal realm: general construction/1

Coda: Given $\mathfrak{g}$ a **k**-Lie algebra (finite or infinite dimensional), which is free as a k-module (k is, as above, a $\mathbb{Q}$-algebra), given any ordered basis $B = (b_i)_{i \in I}$ of $\mathfrak{g}$. As above, for every $\alpha \in \mathbb{N}^{(I)}$, we set

$$B^\alpha = b_{i_1}^{\alpha_1} b_{i_2}^{\alpha_2} \cdots b_{i_m}^{\alpha_m}$$

with $supp(\alpha) = \{i_1 < i_2 < \cdots i_m\}$. We now consider the space

$$\mathcal{A} = span_k\{(B_\alpha) | \alpha \in \mathbb{N}^{(I)}\} \subset \mathcal{U}^*(\mathfrak{g}) \tag{13}$$

It is an convolution subalgebra, due to the formula

$$B_\alpha * B_\beta = \frac{(\alpha+\beta)!}{\alpha!.\beta!}\ B_{\alpha+\beta} \tag{14}$$

Towards the formal realm: general construction/2

Every character χ of $(\mathcal{A}, *, \epsilon)$ can be factored in an MRS way

$$\chi = \prod_{i \in I}^{\rightarrow} e^{\chi(B_{e_i})\, b_i} \tag{15}$$

for the topology of pointwise convergence on $\mathcal{A}$ (k being discrete and the notation of B_α being those of $(\mathrm{Dual} - \mathrm{Basis})$).

Remark. – This formula holds for every character with values in a commutative **k**-algebra $(\mathcal{B}, *_{\mathcal{B}}, 1_{\mathcal{B}})$, in particular with $\mathcal{B} = \mathcal{A}$, one has

$$Id_{\mathcal{U}} = \sum_{\alpha \in \mathbb{N}^{(I)}} B_\alpha \otimes_{Hom} B^\alpha = \prod_{i \in I}^{\rightarrow} e^{B_{e_i} \otimes_{Hom} b_i} \tag{16}$$

where, for $(f, b) \in \mathcal{U}^* \times \mathcal{U}$, $f \otimes_{Hom} b$ stands for $g \in End(\mathcal{U})$ such that $g(x) = f(x).b$.

Concluding remark and final questions

1. A MRS factorization exists with monoids (called free partially commutative, see [7])

$$M(X,\theta) = \langle X; (xy = yx)_{(x,y)\in\theta}\rangle_{\mathbf{Mon}} \tag{17}$$

   where $\theta \subset X \times X$ is a reflexive undirected graph. This is proved using $\mathbf{k}[M(X,\theta)] = \mathcal{U}(Lie_{\mathbf{k}}(X,\theta))$.

2. A unipotent Magnus group with a nice Log-Exp correspondence can be defined for every locally finite monoid. Is there a general MRS factorization ?

3. In the sound cases, what is the combinatorics of different orders ? (Not increasing or decreasing Lyndon words.) Are they useful ?

Thank you for your attention.

Links

1. Categorical framework(s)

   https://ncatlab.org/nlab/show/category
   https://en.wikipedia.org/wiki/Category_(mathematics)

2. Universal problems

   https://ncatlab.org/nlab/show/universal+construction
   https://en.wikipedia.org/wiki/Universal_property

3. Paolo Perrone, *Notes on Category Theory with examples from basic mathematics*, 181p (2020)
   arXiv:1912.10642 [math.CT]

   https://en.wikipedia.org/wiki/Abstract_nonsense

4. Heteromorphism

   https://ncatlab.org/nlab/show/heteromorphism

5. D. Ellerman, *MacLane, Bourbaki, and Adjoints: A Heteromorphic Retrospective*, David EllermanPhilosophy Department, University of California at Riverside

Links/2

6. https://en.wikipedia.org/wiki/Category_of_modules
7. https://ncatlab.org/nlab/show/Grothendieck+group
8. Traces and hilbertian operators
   https://hal.archives-ouvertes.fr/hal-01015295/document
9. State on a star-algebra
   https://ncatlab.org/nlab/show/state+on+a+star-algebra
10. Hilbert module
    https://ncatlab.org/nlab/show/Hilbert+module

[1] N. Bourbaki, *Algebra I (Chapters 1-3)*, Springer 1989.

[2] N. Bourbaki, *Algèbre, Chapitre 8*, Springer, 2012.

[3] N. Bourbaki.– *Lie Groups and Lie Algebras, ch 1-3*, Addison-Wesley, ISBN 0-201-00643-X

[4] P. Cartier, *Jacobiennes généralisées, monodromie unipotente et intégrales itérées*, Séminaire Bourbaki, Volume 30 (1987-1988) , Talk no. 687 , p. 31-52

[5] M. Deneufchâtel, GD, V. Hoang Ngoc Minh and A. I. Solomon, *Independence of Hyperlogarithms over Function Fields via Algebraic Combinatorics*, 4th International Conference on Algebraic Informatics, Linz (2011). Proceedings, Lecture Notes in Computer Science, 6742, Springer.

[6] Jean Dieudonné, *Foundations of Modern Analysis*, Volume 2, Academic Press; 2nd rev edition (January 1, 1969)

[7] G. Duchamp, D.Krob, *Free partially commutative structures*, Journal of Algebra, 156 , 318-359 (1993)

[8] GD, Quoc Huan Ngô and Vincel Hoang Ngoc Minh, *Kleene stars of the plane, polylogarithms and symmetries*, (pp 52-72) TCS 800, 2019, pp 52-72.

[9] GD, Darij Grinberg, Vincel Hoang Ngoc Minh, *Three variations on the linear independence of grouplikes in a coalgebra*, ArXiv:2009.10970 [math.QA] (Wed, 23 Sep 2020)

[10] Gérard H. E. Duchamp, Christophe Tollu, Karol A. Penson and Gleb A. Koshevoy, *Deformations of Algebras: Twisting and Perturbations* , Séminaire Lotharingien de Combinatoire, B62e (2010)

[11] GD, Nguyen Hoang-Nghia, Thomas Krajewski, Adrian Tanasa, *Recipe theorem for the Tutte polynomial for matroids, renormalization group-like approach*, Advances in Applied Mathematics 51 (2013) 345–358.

[12] K.T. Chen, R.H. Fox, R.C. Lyndon, *Free differential calculus, IV. The quotient groups of the lower central series*, Ann. of Math. , 68 (1958) pp. 81–95

[13] V. Drinfel'd, *On quasitriangular quasi-hopf algebra and a group closely connected with Gal($\bar{\mathbb{Q}}/\mathbb{Q}$)*, Leningrad Math. J., 4, 829-860, 1991.

[14] M.E. Hoffman, *Quasi-shuffle algebras and applications*, arXiv preprint arXiv:1805.12464, 2018

[15] H.J. Susmann, *A product expansion for Chen Series*, in Theory and Applications of Nonlinear Control Systems, C.I. Byrns and Lindquist (eds). 323-335, 1986

[16] P. Deligne, *Equations Différentielles à Points Singuliers Réguliers*, Lecture Notes in Math, 163, Springer-Verlag (1970).

[17] M. Lothaire, *Combinatorics on Words*, 2nd Edition, Cambridge Mathematical Library (1997).

[18] Szymon Charzynski and Marek Kus, *Wei-Norman equations for a unitary evolution*, Classical Analysis and ODEs, J. Phys. A: Math. Theor. 46 265208

[19] Rimhac Ree, *Lie Elements and an Algebra Associated With Shuffles*, Annals of Mathematics Second Series, Vol. 68, No. 2 (Sep., 1958)

[20] G. Dattoli, P. Di Lazzaro, and A. Torre, $SU(1,1), SU(2)$, *and* $SU(3)$ *coherence-preserving Hamiltonians and time-ordering techniques. Phys.* Rev. A, 35:1582–1589, 1987.

[21] J. Voight, *Quaternion algebras*, `https://math.dartmouth.edu/~jvoight/quat-book.pdf`

[22] Adjuncts in nlab. `https://ncatlab.org/nlab/show/adjunct`

[23] M. van der Put, M. F. Singer.– *Galois Theory of Linear Differential Equations*, Springer (2003)

[24] Graded rings, see "Graded Rings and Algebras" in
https://en.wikipedia.org/wiki/Graded_ring

[25] How to construct the coproduct of two non-commutative rings
https://math.stackexchange.com/questions/625874

[26] Definition of (commutative) free augmented algebras
https://mathoverflow.net/questions/352726

[27] Closed subgroup (Cartan) theorem without transversality nor Lipschitz condition within Banach algebras
https://mathoverflow.net/questions/356531

[28] Definition of augmented algebras (general)
https://ncatlab.org/nlab/show/augmented+algebra